Absolute instability of a viscous hollow jet

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An investigation of the spatiotemporal stability of hollow jets in unbounded coflowing liquids, using a general dispersion relation previously derived, shows them to be absolutely unstable for all physical values of the Reynolds and Weber numbers. The roots of the symmetry breakdown with respect to the liquid jet case, and the validity of asymptotic models are here studied in detail. Asymptotic analyses for low and high Reynolds numbers are provided, showing that old and well-established limiting dispersion relations [J. W. S. Rayleigh, *The Theory of Sound* (Dover, New York, 1945); S. Chandrasekhar, *Hydrodynamic and Hydromagnetic Stability* (Dover, New York, 1961)] should be used with caution. In the creeping flow limit, the analysis shows that, if the hollow jet is filled with any finite density and viscosity fluid, a steady jet could be made arbitrarily small (compatible with the continuum hypothesis) if the coflowing liquid moves faster than a critical velocity.

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Experience teaches that, contrarily to what occurs in liquid jets, gas jets surrounded by liquid [1–5] are short and extremely prone to local bubbling. Their strong tendency to become absolutely unstable compared to their inverse counterparts—the liquid jets—is well known. The roots of this symmetry breakdown are here analyzed in detail on the basis of a general model [5]. This model is used to analyze the linear convective-absolute (C-A) instability transition of a gas jet in an unbounded coflowing viscous liquid stream with density ρ_l and viscosity μ_l. The asymptotic analysis shows hollow jets to be absolutely unstable for any finite values of the Reynolds (Re) and Weber (We) numbers, in contrast to the liquid jet *in vacuo* limit. Strikingly, however, we found that the universality of the absolutely unstable nature of hollow jets is not predictable using the same asymptotic approximative paths valid for liquid jets in vacuo: [6] our analysis evidences former simplifications asymptotically valid for the liquid jet case to be invalid in the hollow jet case, revealing the roots of breakdown.

Analogously to other paradigmatic results in fluid dynamics (e.g., D'Alembert's paradox [7]), the consideration of an inner fluid with void viscosity μ_g and density ρ_g yields invalid results. Instead, one should carefully access the asymptotic limit from small (but finite) $\mu=\mu_g/\mu_l$ and $\rho=\rho_g/\rho_l$ ratios using a general dispersion relation [5], which takes into account both the viscous and inertial effects of both inner fluid and outer liquid, in either the $\mathrm{Re}\gg 1$ or $\mathrm{Re}\ll 1$ limit. In addition, it is made clear that the old and well-established dispersion relations for either Re=0 or Re=∞ formerly applied to hollow jets ($\rho_g=\mu_g=0$; see Rayleigh [8], Vol. II, p. 362 and Chandrasekhar [9], p. 527, respectively) provide spatiotemporal stability results at radical odds with the asymptotic ones derived from the general dispersion relation [5] for $\rho_g=\mu_g\to 0$, at either vanishing or very large (but finite) Re values, respectively.

To investigate the spatiotemporal stability of the chosen flow configuration, i.e., a cylindrical gas jet, we reduce it to a tractable geometry: an infinite cylindrical jet with radius R_j of a very small density (ρ_g) and viscosity (μ_g) fluid (gas), in an incompressible viscous liquid of density ρ_l viscosity μ_l, and surface tension σ, both coflowing in the z direction with a uniform velocity U relative to the observer. Under the assumption of a small perturbation proportional to $e^{i(kz-\omega t)}$, the conservation equations of mass and momentum of the liquid flow, together with the boundary conditions at both the jet surface (including normal and tangential stress balance) and at infinity, lead to the dispersion relation (DR) between the perturbation wavelength ω and its wave number k derived elsewhere [5,10]. Following the well-established spatiotemporal formalism to describe the absolute-convective character of axisymmetric instabilities in the {Re, We} parametrical space of our problem, we seek for occurrences of $d\omega/dk=0$ in both our general DR [5] and in the simplified one (1) in the lower complex half plane $\mathrm{Im}(k)<0$ [where $\mathrm{Im}(k)$ stands for the imaginary part of k], with $\mathrm{Im}(\omega)\equiv\omega_i\geq 0$ [11–15]. Special precaution has been taken to choose all solutions whose spatial branches departing from the saddle point $d\omega/dk=0$ originate from distinct halves of the k place [[14], p. 484], since these are the only ones providing an absolute growth rate.

To reveal the roots of the symmetry breakdown between liquid jets *in vacuo* and hollow jets, we start from a simplified dispersion relation applicable to hollow jets where ρ_g and μ_g are set equal to zero:

$$k_v^4 - k^4 + 2k^2\frac{K_1(k)}{K_0(k)}\left[2kk_v\frac{K_1'(k_v)}{K_1(k_v)} - (k^2+k_v^2)\frac{K_1'(k)}{K_1(k)}\right] = \frac{\mathrm{Re}^2}{\mathrm{We}}\frac{K_1(k)}{K_0(k)}k(1-k^2). \qquad (1)$$

Here, k_v is defined as $k_v^2=k^2-i\,\mathrm{Re}(\omega-k)$; the Reynolds and Weber numbers are defined as $\mathrm{Re}=\rho_l U R_j/\mu_l$ and $\mathrm{We}=\rho_l U^2 R_j/\sigma$, respectively, and $K_1'(x)=-[K_2(x)+K_0(x)]/2$. The wave frequency ω, time t, wave number k, and streamwise coordinate z scale with U/R_j, R_j/U, $1/R_j$, and R_j, respectively. It is worth noting that the above expression is symmetric with respect to the jet surface to the one used for liquid jets by Leib and Golstein [6]: while they describe the

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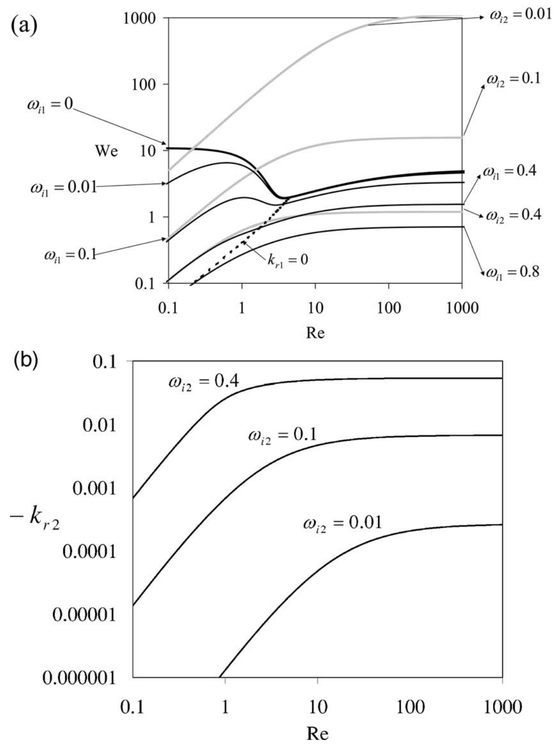

FIG. 1. (a) Solutions of the simplified DR (1) ("hollow" jet case, $\rho=\mu=0$) with $d\omega/dk=0$, $\mathrm{Im}(k)<0$, and $\mathrm{Im}(\omega)\leq 0$, in the $\{\mathrm{Re},\mathrm{We}\}$ plane. Type-1 solutions, (plotted for different values of the growth rate ω_{i1}) are represented by continuous black lines. A thicker black line denotes the theoretical convective-absolute transition ($\omega_{i1}=0$) predicted by this simplified hollow case. Type-2 solutions (for different growth rats ω_{i2}) are represented by light gray lines. (b) Real part of the wave number k_{r2} of type-2 solutions, as a function of the Reynolds number Re.

motion of the inner domain only (liquid jet), expression (1) describes the dynamics of the infinite outer liquid domain around the cylindrical hollow only. It agrees with Chandrasekhar's expression [9] for the liquid jet case surrounded by vacuum [[9], p. 541, Eq. (158)], with two obvious differences: the positive sign at the last summand of the left-hand-side term, and the locus of the singularity of the modified Bessel function ($x\rightarrow\infty$ in the liquid jet case, $x=0$ in the hollow case). This is also a simplified version of the DR obtained by Shen and Li [16], who take into account the density and compressibility of the inner gas but not its viscous effects, of fundamental importance in the creeping flow limit.

Then, it is worth analyzing first the solutions for the simplified DR (1) with $d\omega/dk=0$, $\mathrm{Im}(k)<0$, and $\mathrm{Im}(\omega)\geq 0$, summarized in Fig. 1. This model predicts a C-A instability transition (jetting to bubbling) at moderate We numbers. An interesting feature is the appearance of two families of solutions (types 1 and 2) unfolding from a single one at $\mathrm{Re}\ll 1$ (see Fig. 1, left side of the plot). We should emphasize that these solutions do not necessarily indicate absolutely unstable modes. In fact, solutions with $\mathrm{Real}(k)\equiv k_r<0$ [where $\mathrm{Real}(k)$ stands for the real Part of k] do not satisfy boundary conditions at infinity. Type-1 solutions represent absolutely unstable modes at the right of the line $\mathrm{Real}(k_1)\equiv k_{r1}=0$ only,

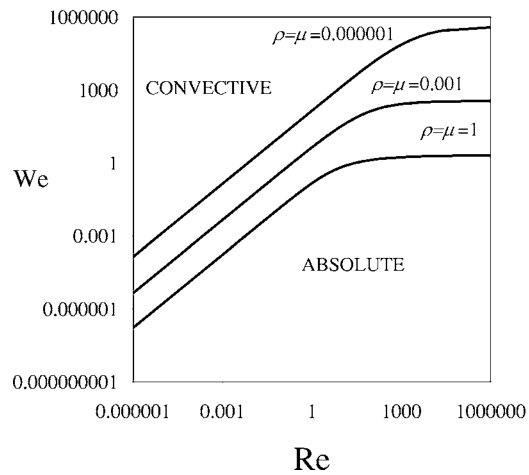

FIG. 2. Loci in the $\{\mathrm{Re},\mathrm{We}\}$ plane of the C-A transition for the general DR [5], for three values of $\rho=\mu\leq 1$. This is an extended (and slightly corrected) version of Fig. 1 in Ref. [5]. Observe that the curves resemble the trends of type-2 solutions (gray lines) in Fig. 1(a).

since k_{r1} is negative to the left (making solutions spurious). The absolute-convective (C-A) instability transition is in this case predicted at the loci indicated by the curve $\omega_{i1}=0$. In addition, all type-2 solutions have $\mathrm{Real}(k_2)\equiv k_{r2}<0$ and are therefore spurious if one strictly clings to the simplified model, even though the $\|k_{r2}\|$ results are very small. We call the attention of the reader to the topology of these type-2 spurious but important solutions (see Fig. 1) for the purposes of this study, since they turn into true absolutely unstable modes when the dynamical effects of the inner fluid are accounted for: since the real parts of the wave number, k_{r2}, are negative but small [see Fig. 1(b)], a commensurately small modification of the model in any direction will drastically modify the convective-absolute nature of the instability transition.

Now, we turn to the use of the general DR [5] which yields the results summarized in Fig. 2: here, the critical We values are plotted as a function of Re for different values of $\rho=\mu$ ($\rho=\rho_g/\rho_l$ and $\mu=\mu_g/\mu_l$). It is worth emphasizing that the parametric range explored in this study for both We and Re span more than 12 orders of magnitude, from very small to very large values. In this study, the inner fluid density and viscosity are smaller than those of the outer liquid, and the hollow jet case is reached when both ρ and μ values vanish. One finds that for arbitrarily small but finite ρ and μ, the relevant "true" absolutely unstable solutions exhibiting a C-A transition [5] come close to those type-2 spurious solutions of the simplified DR (1). Topologically, the Riemann surface of the general DR comes close to that of the simplified DR (1) as ρ and μ vanish, but the asymptotic derivative of the shift appears to be infinity at $\rho=\mu=0$, which produces an abrupt departure of the solutions at that limit. This departure seems to be at the root of the predictive inability of the simplified model, which neglects all dynamical effects coming from the inner fluid. It would also explain why type-2 solutions of the simplified DR appear as spurious, while their immediate "ancestors" from the general DR of the complete model seem to be relevant solutions of the problem. In simpler words, in C-A instability analysis the limits $\rho \to 0$ and $\rho \to 0$ are singular limits for the hollow jet case, as we will explicitly show in the $\mathrm{Re} \gg 1$ and $\mathrm{Re} \ll 1$ cases.

In consequence, the resulting critical We values of the general DR predicting the C-A instability transition are drastically different from the ones corresponding to the simplified one (compare Figs. 1 and 2) since type-1 and type-2 solutions of the latter DR (1) come from very different regions of the Riemann surface (i.e., come from different dynamical balances of the wave modes). Thus, the C-A transition predicted from the simplified model differs abruptly from the asymptotic one predicted from the general DR, the latter valid for all finite Re and We values. In conclusion, one should exercise extraordinary caution at the time of using well-established simplified dispersion relations like those neglecting viscosity effects, in particular to predict C-A transitions.

The limits $\mathrm{Re} \to 0$ and $\mathrm{Re} \to \infty$ represent situations worth analyzing in detail as well, since both foster further drastic simplifications of the general DR. A brief discussion of the validity of these simplifications—as they are established in the literature—is provided below.

The limit Re≫1. Under this assumption, $\|k_v^2\| \sim \|-i\,\mathrm{Re}(\omega - k)\| \gg 1$, which when substituted in Eq. (1) and writing $\omega' = \omega - k$ yields

$$\omega'^2 + \frac{K_1(k)}{\mathrm{We}\,K_0(k)} k(1-k^2) = 0 \tag{2}$$

in accord with Rayleigh [8]. Expression (2) exhibits absolutely unstable solutions for Weber numbers $\mathrm{We} < 5.191\,37$, above which convective instability is predicted. However, a more careful consideration of the solutions of the general [5] DR disclose that $\|\omega\| \ll 1$ and $\|k\| \ll 1$ when $\mathrm{Real}(\omega) \ll 1$, which makes $\|k_v^2\|$ indeterminate for large Re and We values. Thus, predictions of convective instability using DR (2) are invalid, making that very simple model inconsistent for a spatiotemporal analysis. Now, using the general DR, the predicted critical We^* for the C-A instability transition, as a function of the density ratio $\rho = \rho_g/\rho_l$, is asymptotically

$$\mathrm{We}^* = 0.469/\rho \tag{3}$$

for $\mathrm{Re} \gg 1$ and $\rho \ll 1$ (Fig. 2; see also Fig. 1 in Ref. [5]). This makes We^* unbounded for vanishing ρ values, which shows explicitly the singular nature of the hollow jet ($\rho \to 0$) limit. In consequence, by harnessing the general DR through careful asymptotic paths, one concludes that a hollow jet in an infinite inviscid liquid domain is always absolutely unstable.

The limit Re≪1. On the other hand, following the same procedure as in [[9], pp. 527 and 541], one may write the following simplified DR in the limit $\mathrm{Re}=0$:

$$\omega = k + i\frac{1-k^2}{2\mathrm{Ca}\{1+k^2-[kK_0(k)/K_1(k)]^2\}} \tag{4}$$

where $\mathrm{Ca} = \mathrm{We}/\mathrm{Re}$ is the capillary number. This limiting DR does not show solutions of $d\omega/dk = 0$ and $\mathrm{Im}(\omega) \geq 0$ in the lower half plane $\{k\}$ for $\mathrm{Ca} \geq 1$. In sharp contrast, riding on the relevant solutions of the general DR and following a consistent asymptotic path as $\mathrm{Re} \to 0$ (see Fig. 2), one arrives at absolutely unstable asymptotic solutions showing a C-A transition at a critical capillary number Ca^* plotted in Fig. 3 as a function of the viscosity ratio $\mu = \mu_g/\mu_l < 1$. One obtains, for $\mu \ll 1$,

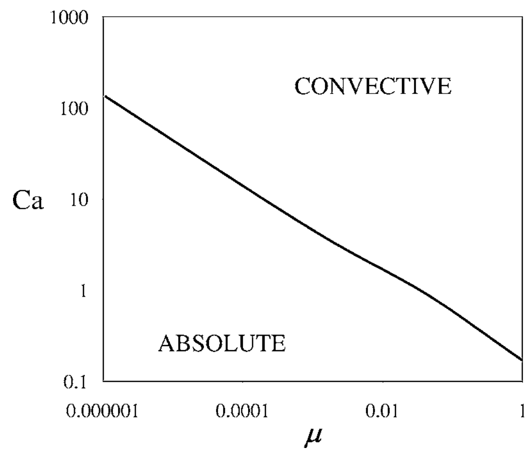

FIG. 3. C-A transition capillary numbers Ca^* as a function of μ (<1), in the limit $\mathrm{Re} \to 0$, from the general DR [10].

$$\mathrm{Ca}^* = 0.139/\mu^{1/2}, \tag{5}$$

again showing the singular nature of the hollow jet case $\mu \to 0$.

This limit expression means that for a vanishing viscosity ratio μ, in the creeping flow limit, the critical capillary number becomes unbounded. However, if the hollow is filled with any tangible fluid (finite ρ and μ values) and the coflowing liquid moves faster than a critical velocity given by

$$U^* = 0.139\sigma(\mu_g\mu_l)^{-1/2}, \tag{6}$$

the jet becomes convectively unstable, which means that the jet becomes stable at a fixed distance from the nozzle independently of the jet radius. This striking result means that, in the creeping flow limit, a steady fluid jet could be made arbitrarily thin ($R_j \to 0$ with $\mathrm{Re} \to 0$, compatible with the continuum hypothesis) in a coflowing liquid if it moves faster than U^*, supporting the existence (stability) of the steady solution obtained by Zhang [17] for a vanishingly small gas spout entrained in a coflowing liquid, a fundamental result that was under controversy [18].

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